Theory and Practice of Artificial Intelligence

Probabilities

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Motivation (Fermat, Pascal)

Consider: pot of money, two-player bet (coin toss)

Problem: game is interrupted — what is fair split of pot *before* coin toss?

Idea:
- assume none of the 2 outcomes of coin toss preferred
- associate a *probability* of $1/2$ with each of the outcomes
- this is the weight by which the payoff of the pot is multiplied for each of the potential outcomes: each of the players gets $1/2$ of the pot

Discussion

Note:

- this is a special case
- in general, the probabilities are not identical for the outcomes
- also, more than 2 outcomes possible

Def.: **random variables** and **probabilities**

- a **random variable** X is an object with potential outcomes $x_1, x_2, \ldots$ from a set $\mathcal{X}$;
- each of these outcomes $x_1, x_2, \ldots$ is associated with a real value, its **probability** $P(X = x_1) \equiv p(x_1), P(X = x_2) \equiv p(x_2), \dots \in \mathbb{R}$ s.t.
  1. $p(x) \geq 0$ for all $x \in \mathcal{X}$;
  2. $\sum_{x \in \mathcal{X}} p(x) = 1$

Example: Die

Consider:

1. die with 6 sides;
2. assume no reason to assume asymmetry, i.e. no side is preferred
   (**Laplace's principle of insufficient reason**)
3. consider outcomes D, probability of each outcome 1,2,...,6 is $P(D=1) = P(D=2) = \cdots = P(D=6) = 1/6$.

Joint Variables

Example: two dice, described by random variables

Joint Variables: (D_1, D_2) with probabilities $p(d_1, d_2) = ?$

Joint Variables/Probabilities

Example: two dice, described by random variables

Joint Variables: (D_1, D_2) with probabilities
$p(d_1, d_2) = 1/36 = 1/6 \cdot 1/6$

Outcomes: all combinations of $\mathcal{D} \times \mathcal{D}$, with $\mathcal{D} = \{1, \ldots, 6\}$
with equal probability

Note: we will see, this is a special case!

Example: Sister/Brother

Consider: random variables $C_1, C_2 \in \{\texttt{boy}, \texttt{girl}\}$, the first and the second child of a family.

Question: assuming that the first child is a girl ($c_1 = \texttt{girl}$), what is the probability that the second child is a boy?

Example: Sister/Brother

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Question: assuming that the first child is a girl ($c_1 = \texttt{girl}$), what is the probability that the second child is a boy?

Answer: consider outcomes:

c_1	c_2	$p(c_1, c_2)$
boy	boy	1/4
boy	girl	1/4
girl	boy	1/4
girl	girl	1/4

Answer: Sister/Brother

Remember Question: **assuming first child is a girl**

Answer: consider outcomes:

c_1	c_2	$p(c_1, c_2)$
boy	boy	1/4
boy	girl	1/4
girl	boy	1/4
girl	girl	1/4

Note: only cases considered with $c_1 = \texttt{girl}$

Here: total weight (probability) of cases with $C_1 = \texttt{girl}$ is 1/2, and 1/4 of which cover the cases where $C_2 = \texttt{boy}$.

Hence: probability that second child is boy if first child is girl given by

$$\frac{1/4}{1/2} = 1/2$$

Example: Sister/Brother II

Question II: assume **one of the children** is a girl, what is the probability that the other child is a boy?

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Answer: consider outcomes:

c_1	c_2	$p(c_1, c_2)$
boy	boy	1/4
boy	girl	1/4
girl	boy	1/4
girl	girl	1/4

Note:
- now total probability of the cases to consider ($C_1 =$ girl or $C_2 =$ girl) is 3/4!
- in 2 of these cases, the other child is a boy, original probability is now 1/4+1/4=1/2

Hence: probability that other child is boy if one child is girl is:

$$\frac{1/2}{3/4} = 2/3$$

Conditional Probabilities

Def.: probability of a random variable Y if another random variable X is given is called **conditional probability**, and written $P(Y = y|X = x) \equiv p(y|x)$.

Example: in boy/girl example, we calculated in

1. Question I: $P(C_2 = \texttt{boy}|C_1 = \texttt{girl})$
2. Question II: $P(C_1 = \texttt{boy}\,\mathit{or}\,C_2 = \texttt{boy}|C_1 = \texttt{girl}\,\mathit{or}\,C_2 = \texttt{girl})$

Joint Probabilities and Marginalization

Summary:

- joint distribution of two variables C_1, C_2: $p(c_1, c_2)$
- probability of only one of the variables obtained by **marginalization** — sum over the other:

$$p(c_1) = \sum_{c_2 \in \mathcal{C}_2} p(c_1, c_2)$$

- example for marginalization (over the second variable): probability of **first child being girl** (was: $1/2$)
- **not marginalization**: probability of **one child being girl** — for this, we had to consider both first and second child!

Conditional Probabilities

Summary:

- conditional distribution of a variable C_2 given another variable C_1: $p(c_2|c_1)$
- example: probability of second child being boy if first child is girl, expressed from the joint distribution:

$$P(C_2 = \texttt{boy}|C_1 = \texttt{girl}) = \frac{P(C_1 = \texttt{girl}, C_2 = \texttt{boy})}{P(C_1 = \texttt{girl})}$$

- note: select all (and only) cases where condition $C_1 = \texttt{girl}$ holds. Take probability of desired case $C_2 = \texttt{boy}$ and normalize by the total probability of the condition $C_1 = \texttt{girl}$.

Conditional Probabilities II

Note: conditional can be computed also in more general cases, namely when outcomes are combined (Question II)
Divide probability of desired outcome through total probability of conditional, in general:

$$p(y|x) = \frac{p(x,y)}{p(x)}$$

Note: in Question II, the notation for the combined conditions (one of the children is a girl/boy) would have to be suitably denoted, but simplified things by summing up the probabilities of all relevant cases.

Bayes' Theorem

Remark: Note that

$$p(d_2|d_1)p(d_1) = p(d_1, d_2) = p(d_1|d_2)p(d_2)\,.$$

From this follows

Bayes' Theorem: One has:

$$p(d_2|d_1) = \frac{p(d_1|d_2)p(d_2)}{p(d_1)}\,.$$

Note:

- Bayes' Theorem highly important: allows one to turn around the direction of a conditional. If conditional in one direction is known, the other can be inferred.
- Sufficient to compute $p(d_1|d_2)p(d_2)$; denominator obtained by normalization.